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Nature of the order-disorder transition in the Vicsek model for the collective motion of self-propelled particles

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One of the most popular approaches to the study of the collective behavior of self-driven individuals is the well-known Vicsek model (VM) [T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, Phys. Rev. Lett. **75**, 1226 (1995)]. In the VM one has that each individual tends to adopt the direction of motion of its neighbors with the perturbation of some noise. For low enough noise the individuals move in an ordered fashion with net transport of mass; however, when the noise is increased, one observes disordered motion in a gaslike scenario. The nature of the order-disorder transition, i.e., first-versus second-order, has originated an ongoing controversy. Here, we analyze the most used variants of the VM unambiguously establishing those that lead either to first- or second-order behavior. By requesting the invariance of the order of the transition upon rotation of the observational frame, we easily identify artifacts due to the interplay between finite-size and boundary conditions, which had erroneously led some authors to observe first-order transitionlike behavior.

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The study and understanding of the collective notion of self-propelled particles is a topic of interdisciplinary interest that has attracted the attention of many scientists for a long time. On the other hand, the ubiquity of the phenomenon and the range of observational scales are striking: it is realized by molecular motors at intracellular level; it occurs when cells move collectively upon tumor growth or wound healing; and of course, it is observed for a large number of living individuals such as amoebae, bacteria, insects, fish, birds, and large quadrupeds [1–8]. Furthermore, another intriguing feature of the phenomenon is the onset of collective order without the presence of any leader, gradient field, or geometrical confinement [9]. Apart from the intrinsic biological interest, technological applications of the knowledge obtained from swarming studies in fields such as robotics, informatics, nanoengineering, and granular material science [10] have also been envisioned. In view of the widespread interest and applications of the phenomenon it is desirable to understand one of the simplest possible models capable of capturing the main features of collective motion in a nontrivial manner. Within this context, Vicsek *et al.* [11] years ago introduced a minimal model [Vicsek model (VM)], which has become the hobbyhorse of statistical physicists aimed to describe a subject that can be technically described as the onset of long-range orientational order via spontaneous symmetry breaking. In the VM, pointlike particles move at fixed velocity trying to align locally with their neighbors, but suffering the presence of some noise. By considering the low-density, low-velocity limits, there is general agreement that the VM exhibits an ordered phase with a macroscopic net mass transport for low noise, while a disordered (gaslike) phase emerges at the high-noise regime. Nevertheless, the nature of that far-from equilibrium phase transition is a matter of open debate. In fact, early simulations by the Vicsek's group are consistent with a continuous second-order transition [11]. This picture has been challenged by the group of Chaté [12] who claims that the transition should be discontinuous, i.e., of first order. The controversy was further stimulated by subsequent papers of Vicsek *et al.* [13,14], Aldana *et al.* [15–17], Dossetti *et al.* [18], and Baglietto *et al.* [19,20], supporting the critical nature of the transition, which are in conflict with the publication of additional results by the group of Chaté [21–23].

In view of the existing dispute, the aim of this Rapid Communication is to provide conclusive evidence on the nature of the phase transition of the VM and some of its variants. The achievement of this goal has become a highly desired prerequisite for further studies in the field and to avoid misunderstandings. A careful reading of the available literature reveals that various authors have introduced subtle changes in the original VM, such as in the velocity update procedure and in the evaluation of the noise. As we will show below, these changes, which are often expected to be irrelevant [12,21,22], are one of the key features for the clarification of the issue. Then, we will first clearly define the most popular variants of the VM, and subsequently, we will present and discuss our simulation results of the simplest imaginable test for the robustness of the phase transition: to check the invariance of the order of the transition under rotation of the observational frame.

Simulations of the VM are performed in dimension $d=2$ where pointlike particles move off-lattice in finite samples of side L with periodic boundary conditions. For the sake of clarity each particle is labeled with an integer index (i), such that its position and velocity are denoted by $\vec{x}_i$ and $\vec{v}_i$, respectively. The direction of motion of the ith particle depends on the average velocity of neighboring particles (including the ith itself) within a circle of radius R. The system is a cellular automaton, so both $\vec{v}_i$ and $\vec{x}_i$ are updated synchronously for $1 \leq i \leq N$, where N is the total number of particles. The absolute value of the velocity of all individuals is assumed to be constant, i.e., v_o, $\forall i$.

Let us now define the most used updating rules:

(1) *Angular noise* (AN). This method, originally proposed by Vicsek *et al.* [11], consists in the determination of the angle of motion of the ith particle as due to the the average

angle of motion of the neighboring j particles, also including the ith itself, which is then affected by the noise term,

$$\theta_i^{t+1} = Arg\left[\sum_{\langle i,j\rangle} e^{i\theta_j^t}\right] + \eta\xi_i^t, \quad (1)$$

where η is the amplitude of the noise, and ξ_i^t is a realization of a δ-correlated white noise uniformly distributed between $-\pi$ and π. Here, the index $\langle i,j\rangle$ in the summations refers to the ith particle and its neighboring j particles. The AN term can be thought as due to the error committed by the particle when trying to adjust its direction of motion to the averaged direction of motion of its neighbors.

(2) *Vectorial noise* (VN). Following Chaté *et al.* [12], one could also argue that the noise arises from each interaction between the ith particle and one of its neighbors. So, instead of Eq. (1) one has

$$\theta_i^{t+1} = Arg\left[\sum_{\langle i,j\rangle} e^{i\theta_j^t} + \eta n_i e^{i\xi_i^t}\right], \quad (2)$$

where n_i is the current number of neighbors of particle i. It is worth mentioning that the magnitude of the AN is independent of the degree of local order, while the VN becomes weaker when the local order is increased.

(3) *Backward update* (BU). By using this schema, originally proposed by Vicsek *et al.* [11], one first evaluates the direction of motion [e.g., with the aid of either Eq. (1) or Eq. (2)] and then proceeds to update the position of the particle according to

$$\vec{x}_i(t+\Delta t) = \vec{x}_i(t) + \vec{v}_i(t)\Delta t. \quad (3)$$

(4) *Forward update* (FU). More recently, various authors [21,22] have adopted this schema that replaces Eq. (3) by

$$\vec{x}_i(t+\Delta t) = \vec{x}_i(t) + \vec{v}_i(t+\Delta t)\Delta t, \quad (4)$$

which is expected to give the same results as in the case of BU [22].

In order to study the phase transition of the VM, the natural order parameter used is the absolute value of the normalized mean velocity (ϕ) given by

$$\phi(t) = \frac{1}{Nv_o}\left|\sum_{i=1}^{N} \vec{v}_i(t)\right|. \quad (5)$$

In all simulations we used $R=1$ and $\Delta t=1$. So, the parameters to be varied are the noise amplitude (η), the particle density ($\rho=N/L^2$), and the velocity v_o.

Figure 1 shows the dependence of ϕ on η as obtained for four different variants of the VM that follow from some specific combinations of the types of noise and update rule, which have been selected as the most studied in the available literature. Figure 1 also shows the plots corresponding to the same variants and parameters, but obtained by using samples where the frame is rotated at random every time step. For this purpose an angle is selected at random and the coordinates of the particles are then transformed accordingly. By using this procedure the configurations of the particles in the sample remain unchanged, but the measurements are

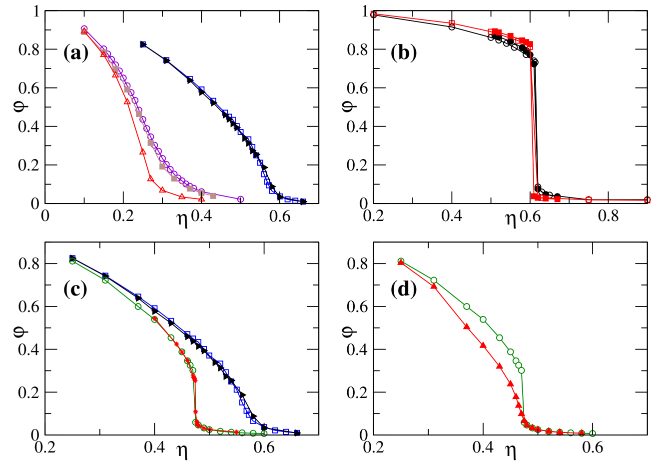

FIG. 1. (Color online) Plots of the order parameter (ϕ) versus the amplitude of the noise (η), as obtained for the VM with different combinations of updating rules. In all cases empty and filled symbols are used for simulations performed in fixed and rotating frames, respectively. (a) The left-hand side and central curves are obtained for $N=32768$, $\rho=0.75$, and $v_o=0.1$, and correspond to the combinations [AN+FU] and [AN+BU], respectively. The right-hand side curves are obtained for $N=131072$, $\rho=2.0$ and $v_o=0.5$, by assuming [AN+BU]. (b) Results obtained for the combination [VN+BU] with $v_o=0.5$, $L=32$, and $N=2048$ ($\rho=2.0$, circles) and $L=4036$ ($\rho=4.0$, squares). (c) and (d) Results corresponding to $N=131072$, $\rho=2$ and $v_o=0.5$, and assuming AN. In (c) the cases of FU (squares) and BU (circles) are compared. Also, the stars in (c) show data estimated by scanning Fig. 1 of Ref. [22]. Furthermore, (d) shows that the combination [AN+FU] is no longer invariant under the rotation of the frame and the transition is actually smooth (triangles) instead of abrupt (circles).

performed from a new (rotated) frame. Since the properties of the VM in general, and the order parameter in particular, are invariant under rotations of the frame, the observance of inconsistencies after this simple test will confirm, beyond any doubt, the existence of artifacts due to the interplay between boundary conditions and finite-size effects. This kind of test is suggested by the observation of the real-time behavior of the system, which for some specific combinations of the rules leads to the formation of bands running along the directions of either the horizontal or vertical axis of the sample, as well as along the principal diagonals (see also Fig. 2). These observations were early reported by Vicsek *et al.* [13] and are dramatically confirmed in many figures of papers published by Chaté *et al.* (see, e.g., Figs. 4, 11, and 1 of Refs. [13,21,22], respectively).

Figure 1(a) corresponds to the VM with two combinations of the rules, namely, [AN+BU] and [AN+FU]. In all cases one observes smooth curves, characteristic of second-order transitions, which are independent of the rotation of the frame. Within the low-density, low-velocity regime [left-hand side and central curves in Fig. 1(a)] we further confirm that the updating rule, i.e., BU versus FU, causes a small shift of the critical point but it does not affect the order of the transition. These results are in agreement with early claims [11] and subsequent results [13] of the group of Vicsek, as well as with our previous combined finite-size scaling and

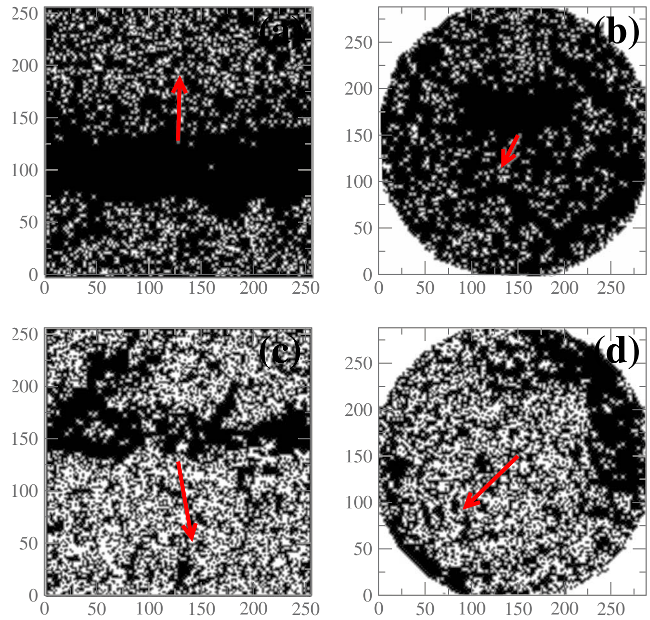

FIG. 2. (Color online) Typical snapshot configurations of the VM with AN, taken within the stationary regime for $N=131072$, $\rho=2$, $v_0=0.5$, $\eta=0.43$, and for different situations as follows: (a) and (c) fixed frame with FU and BU, respectively; (b) and (d) rotating frame with FU and BU, respectively. In all cases the arrows show the average direction of motion of all particles, and its magnitude is proportional to the value of the order parameter. More details in the text.

short-time dynamic simulations [19], which support the critical nature of the transition. Furthermore, all these conclusions are consistent with the results of Aldana *et al.* [16] pointing out that, in the VM and related models, AN leads to the observation of second-order behavior. On the other hand, smooth transitions are also observed for higher velocities and densities [right-hand side curves in Fig. 1(a)]. Figure 1(b) shows that VN leads to the observation of robust (rotationally invariant) first-order behavior. This finding is in agreement which the results of the group of Chaté [12], our own simulations [24], and the claim of Aldana *et al.* [16] stating that the occurrence of first-order transitions in related models is linked to VN.

One of the most intriguing results, however, concerns the combination of [AN+FU], within the high-density, high-velocity regime, as shown in Fig. 1(c). In fact, while [AN +BU] yields second-order behavior, as already discussed within the context of Fig. 1(a) but included in Fig. 1(c) for the sake of comparison, the combination [AN+FU] gives a clear first-order transition. Our results for this latter case are in full agreement with those reported by Chaté (notice that data points scanned from Fig. 1 of Ref. [22] have also been drawn for the sake of comparison). The first conclusion that one can draw from this evidence is that, in contrast to some expectations [21], the update procedure, i.e., FU versus BU, may actually influence the results dramatically. So far, this is not the only unexpected result, but more interesting, the first-order nature claimed for the combination [AN+FU] does not persist when the angle of the frame is randomized at each time step, as shown in Fig. 1(d). So, the first-order nature of the transition is an artifact that can be confirmed straightforwardly by means of a simple test. Further inshight into the origin of this artifact can be gained by analyzing typical snapshot configurations as shown in Fig. 2. Here, all figures correspond to AN but different situations are selected as follows: (i) the upper and lower panels compare FU versus BU, respectively. (ii) The left and right panels compare the case of fixed and randomly rotating frames, respectively. Figure 2(a) was obtained for [AN+FU] with $\eta=0.43$, i.e., a regime within the ordered phase and close to the false coexistence point of the first-order (artifact) transition shown in Figs. 1(c) and 1(d). Here one observes a high-density band moving upwards with an average direction almost parallel to the vertical axis (see the arrow showing the direction of the movement of the center of mass). Also, the order parameter is relatively high ($\varphi \approx 0.4$, see also the magnitude of the arrow that is proportional to φ). This scenario changes dramatically when the observational frame is randomly rotated [Fig. 2(b)]. In this case the largest and denser band becomes almost dissolved [the order parameter decreases significantly in agreement with the results shown in Fig. 1(d)], and the direction of motion is irrelevant. It is worth mentioning that the rotation of the frame causes the particles to almost become confined within a circle of radius $r=L/2$, i.e., naturally replacing the square sample with periodic boundary conditions by a circular geometry where such conditions are irrelevant.

Coming back to Fig. 2(a), one has to recall that due to the periodic boundary conditions the band is actually a "ring" moving in a toroid so that the band is highly correlated along the direction perpendicular to the movement, with a typical correlation length of the order of $\xi \sim L/2$. It is known [13] that for $v_0 > 0.3$, the diffusion of particles in the direction perpendicular to the motion of the system becomes significantly larger than the diffusion in the parallel direction. Then, the flocks tend to elongate laterally until a certain point in which they break into pieces that remain traveling independently. However, when bands grow and reach percolating sizes before disintegration, periodic boundary conditions prevent the natural dismemberment that would take place in an infinite plane, because particles moving away in one sense are approaching in the opposite one. In this way, some order is artificially maintained in this metastable situation. Summing up, one has that the interplay between those mechanisms, the geometry of the sample, and the periodic boundary conditions stabilize a high-density percolating band moving along one of the main directions of the sample. Of course, this situation is changed when considering a rotational frame that destroys the artificial stabilization mechanisms of the band and the pseudo first-order nature of the transition [Fig. 2(b)). On the other hand, on has that the combination [AN+BU] in a fixed frame leads to the formation of loose bands that hardly percolate and the direction of the movement is no longer strictly parallel to one of the main axis of the sample [Fig. 2(c)]. This situation remains almost unchanged upon the rotation of the frame [Fig. 2(d)]. Also notice that in this case the magnitude of the order parameter is maintained, in agreement with the results shown in Fig. 1(c). Here, the second-order nature of the transition is preserved upon the rotation of the frame.

Summing up, we have analyzed the ongoing controversy about the nature of the order-disorder transition of the VM. It is shown that subtle changes introduced to the model, e.g., angular versus vectorial noise, and forward versus backward updates, which are expected to be irrelevant, are actually essential in order to clarify the controversy. We conclude that AN and VN lead to the observation of second- and first-order transitions, irrespective of the update rule.

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